The equation of state for two-dimensional hard-sphere gases: Hard-sphere gases as ideal gases with multi-core boundaries *

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Abstract

The equation of state for a two-dimensional hard-sphere gas is difficult to calculate by usual methods. In this paper we develop an approach for calculating the equation of state of hard-sphere gases, both for two- and three-dimensional cases. By regarding a hard-sphere gas as an ideal gas confined in a container with a multi-core (excluded sphere) boundary, we treat the hard-sphere interaction in an interacting gas as the boundary effect on an ideal quantum gas; this enables us to treat an interacting gas as an ideal one. We calculate the equation of state for a three-dimensional hard-sphere gas with spin j, and compare it with the results obtained by other methods. By this approach the equation of state for a two-dimensional hard-sphere gas can be calculated directly.

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There are various methods for obtaining the equation of state of three-dimensional hard-sphere gases, e.g., the method of pseudopotentials [1], cluster expansions [2, 3]. However, it is difficult to calculate the equation of state of two-dimensional hard-sphere gases. In this paper, for dealing with the two-dimensional hard-sphere gases, we develop an approach for calculating the equation of state of hard-sphere gases, both for two- and three- dimensional cases. This paper falls into three sections. In section 1 we first introduce the basic idea of this approach. For illustrating the approach and demonstrating its validity, we use this approach to calculate the equation of state for three-dimensional hard-sphere gases and compare the result with the standard result given by Lee and Yang [2, 3]. In section 2 we calculate the equation of state of two-dimensional hard-sphere gases. The paper is concluded in section 3.

1 Hard-sphere gases as ideal gases with multi-core boundaries: An approach to two- and three-dimensional interacting gases

By regarding a hard-sphere gas as an ideal gas confined in a container with a multi-core boundary, i.e., a container filled with small excluded spheres (Fig. 1), we calculate the grand potential for two- and three-dimensional interacting gases using the method developed for calculating the boundary effect on an ideal gas. The value of studying hard-sphere gases is: (1) Often a complete knowledge of the detailed interaction potential is not necessary for a satisfactory description of the system, because a particle that is spread out in space sees only an averaged effect of the potential [1]. (2) Experimental progress has been made in recent years in Bose-Einstein condensation in dilute atomic gases [4]; the hard-sphere interaction model is often used as an efficient tool for studying Bose-Einstein condensation, both for dilute atomic gases and liquid helium [5], after Lee and Yang studied pioneered the study of interacting bosons [6]. (3) The hard-sphere gas, as a simplified model, is of great value for investigating the more general theory of interacting gases and can be extended to some more general cases. As a valuable idealized model, the hard-sphere gas has been studied by various theories, such as the method of pseudopotentials [1], cluster expansions [2, 3], and second quantization [7].

The hard-sphere gas is a simplified model of the interacting gas, which replaces the interparticle interaction by the boundary condition the wavefunction $\psi = 0$ on the boundary; such a boundary in the 3N-dimensional configuration space is equivalent to a tree-like hypersurface [1, 8]. In a quantum hard-sphere gas, there are two interplayed effects: the effect of the statistics and the effect of the hard-sphere interaction. The interplay of the effects of statistics and the interparticle interaction often causes difficulties, e.g., it makes the quantum cluster

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expansion considerably involved [9]: to find the l-th virial coefficient for $l > 2$, one has to solve a quantum many-body problem. There are various approaches to hard-sphere gases. The scheme proposed by Lee and Yang, the binary collision method, is to separate out the effect of the statistics [2, 3]; it allows one first takes care of the statistical aspect of the problem and then tackles the dynamical aspect of it [10], and the l-th virial coefficient can be obtained by solving two-body problems. In the method of pseudopotentials the boundary condition is approximately replaced by a non-Hermitian pseudopotential [1, 8], i.e., the boundary condition is converted to a "potential" approximately.

In this paper, by replacing a hard-sphere gas by an ideal quantum gas with a multi-core boundary (Fig. 1), and using the method developed for calculating the boundary effects, we converted the problem of a quantum hard-sphere gas (Fig. 1a) into a problem of an ideal quantum gas confined in a container filled with small excluded spheres (cores) randomly distributed (Fig. 1b). In this approach, roughly speaking, the exchange effect and the effect of classical interaction (in this case, the hard-sphere interaction) are treated separately: the exchange effect on the imperfect gas is embodied in the exchange effect on the ideal gas; the effect of hard-sphere interaction is embodied in the boundary effect. The number of the cores is chosen to equal the total number of particles; the radius of the cores is chosen to equal the diameter of the hard sphere a (the scattering length of the hard sphere) since the area of a core felt by a gas molecule during the scattering process is $4\pi a^2$.

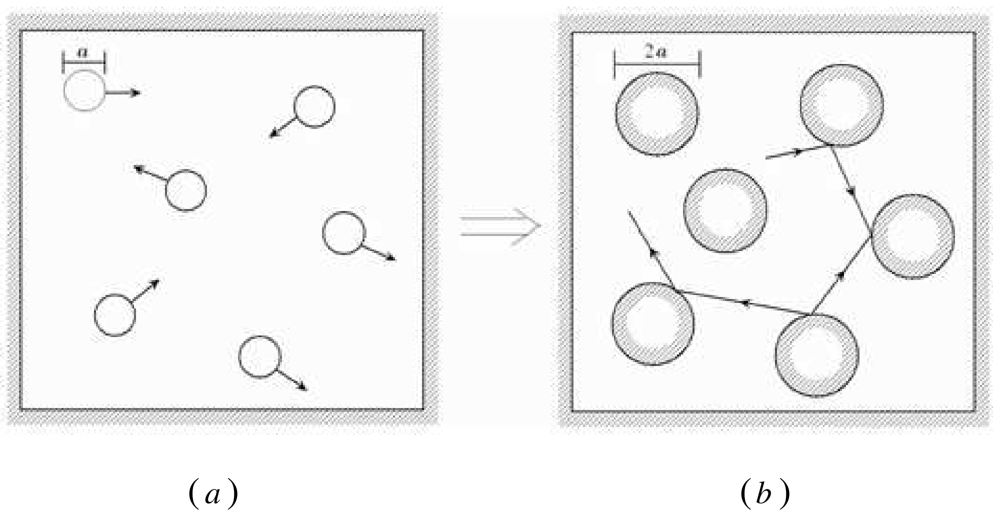

Fig. 1. Regarding a hard-sphere gas as an ideal quantum gas in a multi-core container. (a) A hard-sphere gas in which the diameter of the gas molecule is a. (b) An ideal quantum gas confined in a container with a multi-core boundary. The diameter of the core is $2a$.

Then, the task that to calculate an interacting gas is converted to a task that to calculate the boundary effect on an ideal quantum gas. In preceding papers we developed some methods for calculating boundary effects [11, 12]: one [11], which is based on the mathematical work given by Kac [13], is for two-dimensional cases; the other [12] is for three-dimensional cases. However, to apply these methods to the present problem, in which the boundary is more complex, we need to extend them to more general cases [14]: the grand potentials for ideal quantum gases in three- and two- dimensional confined space can be expressed as

$$\ln \Xi = g\frac{1}{2^0}\frac{\mu_3^o - (-1)^{d-3}\mu_3^i}{\lambda^3}h_{5/2}(z) - g\frac{1}{2^1}\frac{\mu_2^o - (-1)^{d-2}\mu_2^i}{\lambda^2}h_2(z) + g\frac{1}{2^2}\frac{\mu_1^o - (-1)^{d-1}\mu_1^i}{\lambda^1}h_{3/2}(z), \quad (3D) \qquad (1)$$

$$\ln \Xi = g\frac{1}{2^0}\frac{\mu_2^o - (-1)^{d-2}\mu_2^i}{\lambda^2}h_2(z) - g\frac{1}{2^1}\frac{\mu_1^o - (-1)^{d-1}\mu_1^i}{\lambda^1}h_{3/2}(z), \quad (2D) \qquad (2)$$

where d denotes the dimension; $\lambda = h/\sqrt{2\pi mkT}$ is the thermal wavelength, g the number of internal degrees of freedom and for spins $g = 2j+1$, and the function $h_\sigma(z) = \frac{1}{\Gamma(\sigma)}\int_0^\infty \frac{x^{\sigma-1}}{z^{-1}e^x \mp 1}dx$ equals the Bose-Einstein integral $g_\sigma(z)$ or the Fermi-Dirac integral $f_\sigma(z)$ in boson or fermion case respectively. μ_α^o and μ_α^i are the *valuations* of the containers [15] (μ_α^o for outer boundaries and μ_α^i for inner boundaries): in three dimensions for a sphere μ_3 is the volume, μ_2 half the surface area, and μ_1 twice the diameter; in two dimensions, for a disk μ_2 is the area, and μ_1 half the perimeter. However, this result is only valid for spinless distinguishable particles. To take the effects of spin and indistinguishability into account, we need to improve this result one step further.

The hard-sphere interaction is embodied in the boundary terms; the cross-section for collisions between gas molecules is regarded as the surface area of the boundary cores. For distinguishable hard-sphere particles, the cross-section is $4\pi a^2$; however, for indistinguishable ones, the cross-section is $8\pi a^2$, i.e., the contribution from the indistinguishable hard-sphere interaction is twice the contribution from distinguishable ones. Moreover, for the case of particles with spin j, if we include only the s-wave contribution, there are weights $(j+1)/(2j+1)$ for Bosons and $j/(2j+1)$ for Fermions respectively for the unpolarized scattering of identical particles [16]. Thus the grand potential of an ideal quantum gas confined in the container illustrated in Fig. 1b can be written in the following form:

$$\ln \Xi = g\frac{V}{\lambda^3}h_{5/2}(z) + g2\frac{\omega}{2j+1}\left[-\frac{1}{2}\frac{\frac{1}{2}N4\pi a^2}{\lambda^2}h_2(z) + \frac{1}{4}\frac{-2N2a}{\lambda}h_{3/2}(z)\right], \tag{3}$$

where $\omega = j+1$ for bosons and $\omega = j$ for fermions, a is the diameter of the hard sphere and N is the total number of particles in the system. Note that the boundary, of course, includes all cores. Strictly speaking, see Fig. 1, the volume of the system is the volume of the cubical container V minus the total volume of all cores; however, in view of the fact that for a dilute system the former is much larger than the latter and the contribution from the volume of the cores is only of the order $(a/\lambda)^3$, we approximately use V as the volume of the system. In the second term the factor $(1/2)N4\pi a^2$ is half the total surface area of all cores — the second valuation, which is much larger than the magnitude of $V^{2/3}$ in a macroscopic system; in the third term the factor $2\times 2a$ is twice the diameter of a core so $2N2a$ is the third valuation of the boundary (notice that $2N2a \gg V^{1/3}$). The factors 2 and $\omega/(2j+1)$ reflect the effects of indistinguishability and spin. From Eq. (3), the relation $N = z\partial\ln\Xi/\partial z$ gives a differential equation in N. Such an equation for N is solvable (see Appendix); however, its solution is in a complicated form. In the following, we present a series solution, which is easy to be compared with the results given in the literature, and, of course, it can be checked directly that such a series solution is consistent with the solution obtained by directly solving the differential equation for N. For seeking a series solution for N, we express N as a combination of the Bose-Einstein or the Fermi-Dirac integrals, $h_\sigma(z)$,

$$\begin{aligned} N = g\frac{V}{\lambda^3}\Bigg\{ &\left[\sum_\mu A_\mu h_\mu + \sum_{\nu\le\sigma} B_{\nu\sigma}h_\nu h_\sigma + \sum_{\xi\le\eta\le\lambda} C_{\xi\eta\lambda}h_\xi h_\eta h_\lambda + \cdots\right] \\ &+ \frac{a}{\lambda}\left[\sum_\mu A'_\mu h_\mu + \sum_{\nu\le\sigma} B'_{\nu\sigma}h_\nu h_\sigma + \sum_{\xi\le\eta\le\lambda} C'_{\xi\eta\lambda}h_\xi h_\eta h_\lambda + \cdots\right] \\ &+ \left(\frac{a}{\lambda}\right)^2\left[\sum_\mu A''_\mu h_\mu + \sum_{\nu\le\sigma} B''_{\nu\sigma}h_\nu h_\sigma + \sum_{\xi\le\eta\le\lambda} C''_{\xi\eta\lambda}h_\xi h_\eta h_\lambda + \cdots\right] + \cdots\Bigg\}, \end{aligned} \tag{4}$$

where the sum is taken by all possible $h_\sigma(z)$ and the subscripts of the coefficients are taken in ascending order for avoiding repetition, and substituting Eq. (4) into Eq. (3), accurate to $(a/\lambda)^2$, we obtain

$$\begin{aligned} \ln\Xi = g\frac{V}{\lambda^3}\Bigg\{ &h_{5/2} - \frac{a}{\lambda}2g\frac{\omega}{2j+1}\left[\sum_\mu A_\mu h_{3/2}h_\mu + \sum_{\nu\le\sigma} B_{\nu\sigma}h_{3/2}h_\nu h_\sigma + \sum_{\xi\le\eta\le\lambda} C_{\xi\eta\lambda}h_{3/2}h_\xi h_\eta h_\lambda\right] \\ &- \left(\frac{a}{\lambda}\right)^2 2g\frac{\omega}{2j+1}\Bigg[\sum_\mu \pi A_\mu h_2 h_\mu + \sum_{\nu\le\sigma}\pi B_{\nu\sigma}h_2 h_\nu h_\sigma + \sum_{\xi\le\eta\le\lambda}\pi C_{\xi\eta\lambda}h_2 h_\xi h_\eta h_\lambda \\ &- \sum_\mu A'_\mu h_{3/2}h_\mu - \sum_{\nu\le\sigma} B'_{\nu\sigma}h_{3/2}h_\nu h_\sigma - \sum_{\xi\le\eta\le\lambda} C'_{\xi\eta\lambda}h_{3/2}h_\xi h_\eta h_\lambda\Bigg]\Bigg\}. \end{aligned} \tag{5}$$

Deriving N by using the relations $N = z\partial\ln\Xi/\partial z$ and $z\partial h_\sigma/\partial z = h_{\sigma-1}$, after ignoring higher-order terms, we

have

$$N = g\frac{V}{\lambda^3}\Bigg\{h_{3/2}$$

$$-\frac{a}{\lambda}2g\frac{\omega}{2j+1}\left[\sum_{\mu}A_\mu h_{1/2}h_\mu + \sum_{\mu}A_\mu h_{3/2}h_{\mu-1} + \sum_{\nu\leq\sigma}B_{\nu\sigma}h_{1/2}h_\nu h_\sigma + \sum_{\nu\leq\sigma}B_{\nu\sigma}h_{3/2}h_{\nu-1}h_\sigma + \sum_{\nu\leq\sigma}B_{\nu\sigma}h_{3/2}h_\nu h_{\sigma-1}\right]$$

$$-\left(\frac{a}{\lambda}\right)^2 2g\frac{\omega}{2j+1}\Bigg[\sum_{\mu}\pi A_\mu h_1 h_\mu + \sum_{\mu}\pi A_\mu h_2 h_{\mu-1} + \sum_{\nu\leq\sigma}\pi B_{\nu\sigma}h_1 h_\nu h_\sigma + \sum_{\nu\leq\sigma}\pi B_{\nu\sigma}h_2 h_{\nu-1}h_\sigma + \sum_{\nu\leq\sigma}\pi B_{\nu\sigma}h_2 h_\nu h_{\sigma-1}$$

$$+\sum_{\mu}A'_\mu h_{1/2}h_\mu + \sum_{\mu}A'_\mu h_{3/2}h_{\mu-1} + \sum_{\nu\leq\sigma}B'_{\nu\sigma}h_{1/2}h_\nu h_\sigma + \sum_{\nu\leq\sigma}B'_{\nu\sigma}h_{3/2}h_{\nu-1}h_\sigma + \sum_{\nu\leq\sigma}B'_{\nu\sigma}h_{3/2}h_\nu h_{\sigma-1}\Bigg]\Bigg\}$$

and comparing the expression of N with Eq. (4), we obtain the coefficients: $A_{3/2} = 1$, $B'_{1/2,3/2} = -4g\omega/\left(2j+1\right)$, $B''_{1,3/2} = B''_{1/2,2} = -2\pi g\omega/\left(2j+1\right)$, $C''_{1/2,1/2,3/2} = 2C''_{-1/2,3/2,3/2} = 16g^2\left[\omega/\left(2j+1\right)\right]^2$, and the other coefficients equal 0. Then, we achieve the equation of state, Eq. (6)

$$\lambda^3\frac{P}{kT} = \left(2j+1\right)h_{5/2} - 2\omega\left(2j+1\right)\frac{a}{\lambda}h_{3/2}^2 + \left(\frac{a}{\lambda}\right)^2\left[8\omega^2\left(2j+1\right)h_{1/2}h_{3/2}^2 - \omega\left(2j+1\right)2\pi h_{3/2}h_2\right]. \quad (6)$$

Here we have substituted the relation $g = 2j+1$ and replaced the Bose-Einstein integral $g_\sigma\left(z\right)$ by $h_\sigma\left(z\right)$, which includes the Fermi case. This is the result for an ideal gas confined in the container, Fig. 1b, and this is also the result what we want to obtain for a hard-sphere gas, Fig. 1a.

Alternatively, we can also obtain this result in a more rigorous way. For seeking the series solution for N, we can also express N as

$$N = g\frac{V}{\lambda^3}\left[\sum_{\mu=1}^{\infty}A_\mu z^\mu + \frac{a}{\lambda}\sum_{\nu=1}^{\infty}B_\nu z^\nu + \left(\frac{a}{\lambda}\right)^2\sum_{\sigma=1}^{\infty}C_\sigma z^\sigma + \cdots\right]. \quad (7)$$

In the following we take the Bose-Einstein case as an example, and the result for the Fermi-Dirac case can be achieved by a very similar procedure. Substituting Eq. (7) into Eq. (3) and expanding the Bose-Einstein integral as a series by use of $g_\sigma = \sum_{l=1}^{\infty} z^l/l^\sigma$, we have

$$\ln\Xi = g\frac{V}{\lambda^3}\left\{\sum_{l=1}^{\infty}\frac{z^l}{l^{5/2}} - \frac{a}{\lambda}2g\frac{j+1}{2j+1}\left[\sum_{\mu,l=1}^{\infty}\frac{A_\mu}{l^{3/2}}z^{\mu+l} + \frac{a}{\lambda}\sum_{\mu,l=1}^{\infty}\frac{B_\mu}{l^{3/2}}z^{\mu+l} + \left(\frac{a}{\lambda}\right)^2\sum_{\mu,l=1}^{\infty}\frac{C_\mu}{l^{3/2}}z^{\mu+l}\right]\right.$$
$$\left.-\left(\frac{a}{\lambda}\right)^2\pi 2g\frac{j+1}{2j+1}\left[\sum_{\mu,l=1}^{\infty}\frac{A_\mu}{l^2}z^{\mu+l} + \frac{a}{\lambda}\sum_{\mu,l=1}^{\infty}\frac{B_\mu}{l^2}z^{\mu+l} + \left(\frac{a}{\lambda}\right)^2\sum_{\mu,l=1}^{\infty}\frac{C_\mu}{l^2}z^{\mu+l}\right]\right\}. \quad (8)$$

The relation $N = z\partial\ln\Xi/\partial z$, after ignoring higher-order terms, gives

$$N = g\frac{V}{\lambda^3}\left\{\sum_{l=1}^{\infty}l\frac{z^l}{l^{5/2}} - 2g\frac{j+1}{2j+1}\frac{a}{\lambda}\sum_{\mu,l=1}^{\infty}\left(\mu+l\right)\frac{A_\mu}{l^{3/2}}z^{\mu+l} - 2g\frac{j+1}{2j+1}\left(\frac{a}{\lambda}\right)^2\sum_{\mu,l=1}^{\infty}\left(\mu+l\right)\frac{B_\mu}{l^{3/2}}z^{\mu+l}\right.$$
$$\left.-2\pi g\frac{j+1}{2j+1}\left(\frac{a}{\lambda}\right)^2\sum_{\mu,l=1}^{\infty}\left(\mu+l\right)\frac{A_\mu}{l^2}z^{\mu+l}\right\}$$
$$= g\frac{V}{\lambda^3}\left\{\sum_{\mu=1}^{\infty}\frac{z^\mu}{\mu^{3/2}} - 2g\frac{j+1}{2j+1}\left[\frac{a}{\lambda}\sum_{\mu=2}^{\infty}\sum_{k=1}^{\mu-1}\frac{\mu A_k}{\left(\mu-k\right)^{3/2}}z^\mu - \left(\frac{a}{\lambda}\right)^2\sum_{\mu=2}^{\infty}\sum_{k=1}^{\mu-1}\left(\frac{-\mu\pi A_k}{\left(\mu-k\right)^2} - \frac{\mu B_k}{\left(\mu-k\right)^{3/2}}\right)z^\mu\right]\right\}.$$

Comparing with Eq. (7), we obtain

$$A_\mu = \frac{1}{\mu^{3/2}}, \quad B_\mu = -2g\frac{j+1}{2j+1}\sum_{k=1}^{\mu-1}\frac{\mu}{\left[\left(\mu-k\right)k\right]^{3/2}},$$

$$C_\mu = 2g\frac{j+1}{2j+1}\left[\sum_{k=1}^{\mu-1}\frac{-\pi\mu}{\left(\mu-k\right)^2 k^{3/2}} + 2g\frac{j+1}{2j+1}\sum_{k=2}^{\mu-1}\sum_{l=1}^{k-1}\frac{\mu}{\left(\mu-k\right)^{3/2}}\frac{k}{\left[\left(k-l\right)l\right]^{3/2}}\right],$$

where the summation $\sum_a^b$ should be regarded as 0 when $b<a$. Substituting these parameters into Eq. (8) gives

$$\ln\Xi = g\frac{V}{\lambda^3}\left\{\sum_{l=1}^{\infty}\frac{z^l}{l^{5/2}} + 2g\frac{j+1}{2j+1}\left[-\frac{a}{\lambda}\sum_{\mu,l=1}^{\infty}\frac{1}{l^{3/2}}\frac{1}{\mu^{3/2}}z^{\mu+l}\right.\right.$$
$$\left.\left. + \left(\frac{a}{\lambda}\right)^2\left(2g\frac{j+1}{2j+1}2\sum_{\mu,l,k=1}^{\infty}\frac{1}{l^{3/2}}\frac{1}{\mu^{1/2}}\frac{1}{k^{3/2}}z^{\mu+k+l} - \pi\sum_{\mu,l=1}^{\infty}\frac{1}{l^2}\frac{1}{\mu^{3/2}}z^{\mu+l}\right)\right]\right\}.$$

Performing the summations and substituting the relation $g=2j+1$, we again achieve the equation of state, Eq. (6).

Our result Eq. (6) compares well with the result based on the binary collision method given by Lee and Yang [3],

$$\lambda^3\frac{P}{kT} = (2j+1)\,h_{5/2} - 2\omega\,(2j+1)\,\frac{a}{\lambda}h_{3/2}^2 + \left(\frac{a}{\lambda}\right)^2\left[8\omega^2\,(2j+1)\,h_{1/2}h_{3/2}^2 + \omega\,(2j+1)\,8F\,(\pm z)\right], \tag{9}$$

where $F\,(z) = \sum_{r,s,t=1}^{\infty}(rst)^{-1/2}(r+s)^{-1}(r+t)^{-1}z^{r+s+t}$, "+" for bosons and "−" for fermions. The first-order contributions, a/λ, are the same; the difference only appears in the second term of the second-order, $(a/\lambda)^2$, contribution. To show this difference clearly, as an example, we compare them in the low-temperature and high-density Fermi case, which is the most interesting case — the quantum case. The asymptotic forms for large z are $8F\,(-z)\sim -2.10\,(\ln z)^{7/2}$ and $-2\pi h_{3/2}h_2 \sim -2.36\,(\ln z)^{7/2}$. For Bose cases, at the high temperatures and low temperatures, our result is consistent with Lee and Yang's. At low temperatures and high densities, it is physically meaningless to compare the results Eqs. (6) and (9) since they are both invalid due to the existence of the critical point; in fact in this case the perturbation theory in a makes no sense [6, 17]. Notice that the methods given in Ref. [3] and in this paper are both approximate methods giving fugacity series of grand potentials.

2 The equation of state for two-dimensional hard sphere gases

The two-dimensional hard-sphere quantum gases which are difficult to calculate by usual methods can also be calculated by this approach. Like the three-dimensional case, the grand potential for the two-dimensional cases can be expressed as

$$\begin{aligned}\ln\Xi &= g\frac{1}{\lambda^2}\left(S - 2\frac{\omega}{2j+1}N\pi a^2\right)h_2 - g\frac{1}{2}\frac{1}{\lambda}\left(2\frac{\omega}{2j+1}N\frac{1}{2}2\pi a\right)h_{3/2}\\ &= g\frac{S}{\lambda^2}h_2 - g2\frac{\omega}{2j+1}\frac{a}{\lambda}\frac{\pi}{2}Nh_{3/2} - g2\frac{\omega}{2j+1}\left(\frac{a}{\lambda}\right)^2\pi N h_2.\end{aligned} \tag{10}$$

It should be emphasized that, in the two-dimensional case we take the area of the cores into account since it provides a second-order, $(a/\lambda)^2$, contribution (Recall that in the three-dimensional case we ignore the contribution coming from the volume of the cores since the volume contribution is of the order $(a/\lambda)^3$).

By analyzing the grand potential Eq. (10), we express the particle number N as

$$\begin{aligned}N = g\frac{S}{\lambda^2}\Bigg\{&\left[\sum_{\mu}A_\mu h_\mu + \sum_{\nu\le\sigma}B_{\nu\sigma}h_\nu h_\sigma + \sum_{\xi\le\eta\le\lambda}C_{\xi\eta\lambda}h_\xi h_\eta h_\lambda + \cdots\right]\\ &+\frac{a}{\lambda}\left[\sum_{\mu}A'_\mu h_\mu + \sum_{\nu\le\sigma}B'_{\nu\sigma}h_\nu h_\sigma + \sum_{\xi\le\eta\le\lambda}C'_{\xi\eta\lambda}h_\xi h_\eta h_\lambda + \cdots\right]\\ &+\left(\frac{a}{\lambda}\right)^2\left[\sum_{\mu}A''_\mu h_\mu + \sum_{\nu\le\sigma}B''_{\nu\sigma}h_\nu h_\sigma + \sum_{\xi\le\eta\le\lambda}C''_{\xi\eta\lambda}h_\xi h_\eta h_\lambda + \cdots\right] + \cdots\Bigg\}.\end{aligned} \tag{11}$$

Substituting N into Eq. (10), we have

$$\ln\Xi = g\frac{S}{\lambda^2}h_2 - g\frac{S}{\lambda^2}\pi g\frac{\omega}{2j+1}\frac{a}{\lambda}\left[\sum_{\mu}A_\mu h_{3/2}h_\mu + \sum_{\nu\leq\sigma}B_{\nu\sigma}h_{3/2}h_\nu h_\sigma + \sum_{\xi\leq\eta\leq\lambda}C_{\xi\eta\lambda}h_{3/2}h_\xi h_\eta h_\lambda\right]$$
$$- g\frac{S}{\lambda^2}\pi g\frac{\omega}{2j+1}\left(\frac{a}{\lambda}\right)^2\left[\sum_{\mu}A'_\mu h_{3/2}h_\mu + \sum_{\nu\leq\sigma}B'_{\nu\sigma}h_{3/2}h_\nu h_\sigma + \sum_{\xi\leq\eta\leq\lambda}C'_{\xi\eta\lambda}h_{3/2}h_\xi h_\eta h_\lambda\right]$$
$$- g\frac{S}{\lambda^2}2\pi g\frac{\omega}{2j+1}\left(\frac{a}{\lambda}\right)^2\left[\sum_{\mu}A_\mu h_2 h_\mu + \sum_{\nu\leq\sigma}B_{\nu\sigma}h_2 h_\nu h_\sigma + \sum_{\xi\leq\eta\leq\lambda}C_{\xi\eta\lambda}h_2 h_\xi h_\eta h_\lambda\right]. \tag{12}$$

By $N = z\partial\ln\Xi/\partial z$, after Ignoring higher-order terms, we have

$$N = g\frac{S}{\lambda^2}h_1$$
$$- g\frac{S}{\lambda^2}\pi g\frac{\omega}{2j+1}\frac{a}{\lambda}\left[\sum_{\mu}A_\mu h_{1/2}h_\mu + \sum_{\mu}A_\mu h_{3/2}h_{\mu-1} + \sum_{\nu\leq\sigma}B_{\nu\sigma}h_{1/2}h_\nu h_\sigma + \sum_{\nu\leq\sigma}B_{\nu\sigma}h_{3/2}h_{\nu-1}h_\sigma + \sum_{\nu\leq\sigma}B_{\nu\sigma}h_{3/2}h_\nu h_{\sigma-1}\right]$$
$$- g\frac{S}{\lambda^2}\pi g\frac{\omega}{2j+1}\left(\frac{a}{\lambda}\right)^2\left[\sum_{\mu}A'_\mu h_{1/2}h_\mu + \sum_{\mu}A'_\mu h_{3/2}h_{\mu-1} + \sum_{\nu\leq\sigma}B'_{\nu\sigma}h_{1/2}h_\nu h_\sigma + \sum_{\nu\leq\sigma}B'_{\nu\sigma}h_{3/2}h_{\nu-1}h_\sigma + \sum_{\nu\leq\sigma}B'_{\nu\sigma}h_{3/2}h_\nu h_{\sigma-1}\right]$$
$$- g\frac{S}{\lambda^2}2\pi g\frac{\omega}{2j+1}\left(\frac{a}{\lambda}\right)^2\left[\sum_{\mu}A_\mu h_1 h_\mu + \sum_{\mu}A_\mu h_2 h_{\mu-1} + \sum_{\nu\leq\sigma}B_{\nu\sigma}h_1 h_\nu h_\sigma + \sum_{\nu\leq\sigma}B_{\nu\sigma}h_2 h_{\nu-1}h_\sigma + \sum_{\nu\leq\sigma}B_{\nu\sigma}h_2 h_\nu h_{\sigma-1}\right].$$

Comparing the coefficients with the corresponding coefficients in (11), we have

$$A_1 = 1, A_\mu = 0, B_{\nu\sigma} = 0, C_{\xi\eta\lambda} = 0$$
$$A'_\mu = 0, B'_{1/2,1} = -\pi g\frac{\omega}{2j+1}, B'_{0,3/2} = -\pi g\frac{\omega}{2j+1}, B'_{\nu\sigma} = 0, C'_{\xi\eta\lambda} = 0$$
$$A''_\mu = 0, B''_{1,1} = -2\pi g\frac{\omega}{2j+1}, B''_{0,2} = -2\pi g\frac{\omega}{2j+1}, B''_{\nu\sigma} = 0,$$
$$C''_{1/2,1/2,1} = \left(\pi g\frac{\omega}{2j+1}\right)^2, C''_{0,1/2,3/2} = 3\left(\pi g\frac{\omega}{2j+1}\right)^2, C''_{-1/2,1,3/2} = \left(\pi g\frac{\omega}{2j+1}\right)^2,$$
$$C''_{-1,3/2,3/2} = \left(\pi g\frac{\omega}{2j+1}\right)^2, C''_{\xi\eta\lambda} = 0.$$

Then we obtain the equation of state for two-dimensional gases ($g = 2j+1$):

$$\lambda^2\frac{P}{kT} = (2j+1)\,h_2 + \omega\,(2j+1)\left[-\frac{a}{\lambda}\pi h_1 h_{3/2} + \left(\frac{a}{\lambda}\right)^2\left(\omega\pi^2 h_0 h_{3/2}^2 + \omega\pi^2 h_{1/2}h_1 h_{3/2} - 2\pi h_1 h_2\right)\right]. \tag{13}$$

Alternatively we can also obtain the equation of state in a more rigorous way. For seeking a series solution for N, we expand N as

$$N = g\frac{S}{\lambda^2}\left[\sum_{\mu}A_\mu z^\mu + \frac{a}{\lambda}\sum_{\mu}B_\mu z^\mu + \left(\frac{a}{\lambda}\right)^2\sum_{\mu}C_\mu z^\mu + \cdots\right]. \tag{14}$$

Substituting N into Eq. (10), taking the Bose case as an example, we have ($\omega = j+1$)

$$\ln\Xi = g\frac{S}{\lambda^2}\sum_{l=1}^{\infty}\frac{z^l}{l^2} - g\frac{S}{\lambda^2}\frac{a}{\lambda}\left(\pi g\frac{j+1}{2j+1}\right)\sum_{\mu,l=1}^{\infty}A_\mu\frac{z^{\mu+l}}{l^{3/2}}$$
$$- g\frac{S}{\lambda^2}\left(\frac{a}{\lambda}\right)^2\left(\pi g\frac{j+1}{2j+1}\right)\sum_{\mu,l=1}^{\infty}\left[B_\mu\frac{z^{\mu+l}}{l^{3/2}} + 2A_\mu\frac{z^{\mu+l}}{l^2}\right]$$
$$= g\frac{S}{\lambda^2}\sum_{k=1}^{\infty}\frac{z^k}{k^2} - g\frac{S}{\lambda^2}\frac{a}{\lambda}\pi g\frac{j+1}{2j+1}\sum_{k=2}\sum_{\mu=1}^{k-1}A_\mu\frac{z^k}{(k-\mu)^{3/2}}$$
$$- g\frac{S}{\lambda^2}\left(\frac{a}{\lambda}\right)^2\pi g\frac{j+1}{2j+1}\sum_{k=2}\sum_{\mu=1}^{k-1}\left[B_\mu\frac{1}{(k-\mu)^{3/2}} + 2A_\mu\frac{1}{(k-\mu)^2}\right]z^k. \tag{15}$$

Then we can calculate N:

$$N = g\frac{S}{\lambda^2}\sum_{k=1}^{\infty}\frac{z^k}{k} - g\frac{S}{\lambda^2}\frac{a}{\lambda}\pi g\frac{j+1}{2j+1}\sum_{k=2}\sum_{\mu=1}^{k-1} kA_\mu \frac{z^k}{(k-\mu)^{3/2}}$$
$$- g\frac{S}{\lambda^2}\left(\frac{a}{\lambda}\right)^2 \pi g\frac{j+1}{2j+1}\sum_{k=2}\sum_{\mu=1}^{k-1} k\left[B_\mu\frac{1}{(k-\mu)^{3/2}} + 2A_\mu\frac{1}{(k-\mu)^2}\right] z^k. \tag{16}$$

Comparing with (14) we obtain the coefficients:

$$A_k = \frac{1}{k}$$
$$B_1 = 0,$$
$$B_k = -\pi g\frac{j+1}{2j+1}\sum_{\mu=1}^{k-1}\frac{k}{\mu}\frac{1}{(k-\mu)^{3/2}}$$
$$C_1 = 0,$$
$$C_k = \left(\pi g\frac{j+1}{2j+1}\right)^2\sum_{\mu=2}^{k-1}\sum_{\nu=1}^{\mu-1}\frac{k\mu}{\nu}\frac{1}{(\mu-\nu)^{3/2}}\frac{1}{(k-\mu)^{3/2}} - 2\pi g\frac{j+1}{2j+1}\sum_{\mu=1}^{k-1}\frac{k}{\mu}\frac{1}{(k-\mu)^2}$$

Substituting these coefficients into Eq. (15), we have

$$\ln\Xi = g\frac{S}{\lambda^2}\sum_{l=1}^{\infty}\frac{z^l}{l^2} - g\frac{S}{\lambda^2}\frac{a}{\lambda}\left(g\pi\frac{j+1}{2j+1}\right)\sum_{\mu,l=1}^{\infty}\frac{1}{\mu}\frac{z^{\mu+l}}{l^{3/2}}$$
$$+ g\frac{S}{\lambda^2}\left(\frac{a}{\lambda}\right)^2\left(\pi g\frac{j+1}{2j+1}\right)^2\sum_{k,l,\nu=1}^{\infty}\frac{\nu+k}{\nu}\frac{1}{k^{3/2}}\frac{z^{\nu+k+l}}{l^{3/2}} - g\frac{S}{\lambda^2}\left(\frac{a}{\lambda}\right)^2\left(2\pi g\frac{j+1}{2j+1}\right)\sum_{\mu,l=1}^{\infty}\frac{1}{\mu}\frac{z^{\mu+l}}{l^2}$$

Therefore, following a similar procedure to that which led to Eq. (6), we obtain the equation of state for the two-dimensional hard-sphere gas, Eq. (13), again.

3 Conclusions

In conclusion, by using a method developed for calculating the boundary effect we treat a hard-sphere gas as an ideal quantum gas in a multi-core container, and calculate the grand potentials for hard-sphere gases in two and three dimensions. The result shows that by only replacing the moving gas molecules with a fixed boundary, we can obtain the results almost accurate to the second order, $(a/\lambda)^2$. This is because in this approach we use the s-wave scattering cross-section as the area of the cores. When the p-wave contribution is taken into account, one should consider the effect of the motion of the cores. Such a simplified model for interacting gases allows us to calculate the hard-sphere gas in a simple manner, and can be used to calculate the boundary effects in interacting quantum gases since in this treatment the hard-sphere interaction is regarded as a boundary effect, which we will discuss elsewhere. One advantage of this approach is that it can be used to calculate the hard-sphere gas confined in a container with the Dirichlet boundary condition since in this paper the method used to calculate the boundary effect is valid both for periodic and Dirichlet boundary conditions; while, e.g., in the pseudopotential method the periodic boundary play an essential role in the calculation since with such a boundary condition the momentum is conserved [8]. For seeking more higher-order solutions, we need to take into account other corrections, e.g., the correction caused by the movement of the cores, etc. In such a consideration the grand potential cannot be expanded in powers of a/λ as the higher-order result obtained by using the method of pseudopotentials [18]. This method can be extended to more general cases: by replacing the diameter of hard-spheres with the scattering length, we can obtain a generalized result which can be applied to other kinds of interacting gases. Moreover, this result still can be carried one step further. The expansion coefficients can be explained as form factors. In the case of hard-sphere interactions, such form factors are constants. If we introduce a set of parameter-dependent form factors, the method can be generalized to describe more complex interacting gases. Such a treatment, which will be discussed in detail elsewhere, is equivalent to replacing the hard sphere by a soft sphere which has inner structures. Moreover, this approach can also be directly applied to quantum gases or liquids in porous media. Porous media can be regarded as containers with many holes (cores); in this case the particle number N in Eq. (3) should be replaced by the number of such holes (cores) which are actually real boundaries of the container.

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4 Appendix[19]

As mentioned above, the equation for N is solvable; in this Appendix we give the solution for this equation. From Eq. (3), the relation $N = z\partial \ln \Xi/\partial z$ gives a differential equation for N,

$$\frac{\partial N}{\partial z}+\frac{1+\omega\frac{2a}{\lambda}h_{1/2}+\omega\frac{2\pi a^2}{\lambda^2}h_1}{\omega z\left(\frac{2a}{\lambda}h_{3/2}+\frac{2\pi a^2}{\lambda^2}h_2\right)}N-\frac{(2j+1)\frac{V}{\lambda^3}h_{3/2}}{\omega z\left(\frac{2a}{\lambda}h_{3/2}+\frac{2\pi a^2}{\lambda^2}h_2\right)}=0. \tag{17}$$

The solution of this equation is

$$N=-e^{\int -p(z)dz}\left[\int e^{\int p(z)dz}q(z)\,dz\right], \tag{18}$$

where

$$p(z)=\frac{1+\omega\frac{2a}{\lambda}h_{1/2}+\omega\frac{2\pi a^2}{\lambda^2}h_1}{\omega z\left(\frac{2a}{\lambda}h_{3/2}+\frac{2\pi a^2}{\lambda^2}h_2\right)},$$
$$q(z)=-\frac{(2j+1)\frac{V}{\lambda^3}h_{3/2}}{\omega z\left(\frac{2a}{\lambda}h_{3/2}+\frac{2\pi a^2}{\lambda^2}h_2\right)}.$$

However the form of this solution is complex, and it is difficult to covert this solution to a simple and more useful form. Of course, the solution given by Eq. (18) is consistent with the series solution given in the paper. To check this, we can, for example, expand Eq. (18) and compare it with the expansion of the result given in the paper. When $z < 1$, the expansion of Eq. (18) and the result of N in the paper both are

$$N=(2j+1)\frac{V}{\lambda^3}z+(2j+1)\frac{V}{\lambda^3}\left[\frac{\sqrt{2}}{4}-4\omega\frac{a}{\lambda}-4\pi\omega\left(\frac{a}{\lambda}\right)^2\right]z^2+\cdots.$$

They are the same. In a word, what we do in the paper is to find a series solution for the differential equation of N.

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[19] We would like to thank the anonymous referee of our paper *Europhys. Lett.*, **72** (6), pp. 887–893 (2005). Before he or she helpfully points out that the differential equation of N is solvable, we do not know this due to the complex form of this equation.